A BOHL–BOHR–KADETS TYPE THEOREM CHARACTERIZING BANACH SPACES NOT CONTAINING c_0

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Abstract. We prove that a separable Banach space E does not contain a copy of the space c_0 of null-sequences if and only if for every doubly power-bounded operator T on E and for every vector $x \in E$ the relative compactness of the sets $\{T^{n+m}x - T^n x : n \in \mathbb{N}\}$ (for some/all $m \in \mathbb{N}$, $m \geq 1$) and $\{T^n x : n \in \mathbb{N}\}$ are equivalent. With the help of the Jacobs–de Leeuw–Glicksberg decomposition of strongly compact semigroups the case of (not necessarily invertible) power-bounded operators is also handled.

This note concerns the following problem: Given a Banach space E, a bounded linear operator $T \in \mathscr{L}(E)$ and a vector $x \in E$, we would like to conclude the relative compactness of the orbit

$$\big\{T^n x : \ n \in \mathbb{N}\big\} \subseteq E$$

from the relative compactness of the consecutive differences of the iterates

$$\big\{T^{n+1}x - T^n x : \ n \in \mathbb{N}\big\} \subseteq E.$$

This problem is a discrete, "linear operator analogue" of the classical Bohl–Bohr theorem about the integration of almost periodic functions. Before going to the results let us explain this connection.

Given a (Bohr) almost periodic function $f : \mathbb{R} \to \mathbb{C}$ with its integral $F(t) = \int_0^t f(s)\mathrm{d}s$ bounded, then F is almost periodic itself. This result was extended to Banach space valued almost periodic functions $f : \mathbb{R} \to E$ by M. I. Kadets [7], provided that E does not contain an isomorphic copy of the Banach space c_0 of null-sequences. Actually, the validity of this integration result for *every* almost periodic function $f : \mathbb{R} \to E$ characterizes the absence of c_0 in the Banach space E.

The generalization of Kadets' result—which explains the connection to our problem—was studied by Basit for functions $f : G \to E$ defined on a group G and taking values in the Banach space E (for simplicity suppose now G to be Abelian). In [2] Basit proved that if $F : G \to E$ is a bounded function with almost periodic difference functions

$$F(\cdot + g) - F(\cdot) \quad \text{for all } g \in G,$$

and E does not contain c_0, then F is almost periodic. The relation to Kadets' result is the following: If $f : \mathbb{R} \to E$ is almost periodic, so is $F_\varepsilon(t) := \int_t^{t+\varepsilon} f(s)\mathrm{d}s$

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for every $\varepsilon > 0$. So Kadets' theorem tells that if $F(t) = \int_0^t f(s)\mathrm{d}s$ is bounded and E does not contain c_0, then the almost periodicity of

$$F(\cdot + \varepsilon) - F(\cdot) \quad \text{for all } \varepsilon > 0$$

implies that of F.

Now returning to our problem, suppose $T \in \mathscr{L}(E)$ is *doubly power-bounded* (i.e., $T, T^{-1} \in \mathscr{L}(E)$ are both power-bounded). Then by applying Basit's result to the function $F : \mathbb{Z} \to E$, $F(n) := T^n x$, we obtain that $\{T^n x : n \in \mathbb{Z}\}$ is relatively compact in E if

$$\big\{T^{n+m}x - T^n x : n \in \mathbb{Z}\big\} \quad \text{is relatively compact for all } m \in \mathbb{Z},$$

for which it suffices that

$$\big\{T^{n+1}x - T^n x : n \in \mathbb{Z}\big\} \quad \text{is relatively compact.}$$

Let us record this latter fact in the next lemma.

Lemma 1. *Let E be a Banach space and $T \in \mathscr{L}(E)$ be a power-bounded operator. If for some $x \in E$ the set*

$$\big\{T^{n+1}x - T^n x : n \in \mathbb{N}\big\}$$

is relatively compact, then so is the set

$$\big\{T^{n+m}x - T^n x : n \in \mathbb{N}\big\}$$

for all $m \in \mathbb{N}$.

Proof. Denote by D_1 the first set and by D_m the second one. We may suppose $m \geq 2$. By continuity of T the sets TD_1, $T^2 D_1$, $T^{m-1}D_1$ are all relatively compact. Since

$$T^{m+n}x - T^n x = T^{m-1}(T^n x - x) + T^{m-2}(T^n x - x) + \cdots + T(T^n x - x) + (T^n x - x),$$

we obtain $D_m \subseteq TD_1 + T^2 D_1 + \cdots + T^{m-1}D_1$ implying the relative compactness of D_m. □

So our problem can be answered satisfactorily for doubly power-bounded operators on Banach spaces not containing c_0. The situation is different if T is non-invertible, or invertible but with not power-bounded inverse. To enlighten what may be true in such a situation, let us recall a result of Ruess and Summers, who considered the generalization of the Bohl–Bohr–Kadets result for functions $f : \mathbb{R}_+ \to E$, see [8, Thm. 2.2.2] or [9, Thm. 4.3].

Given an *asymptotically almost periodic* function $f : \mathbb{R}_+ \to E$, one can find an *almost periodic* one $f_\mathrm{r} : \mathbb{R} \to E$ and another function $f_\mathrm{s} : \mathbb{R}_+ \to E$ vanishing at infinity such that $f = f_\mathrm{s} + f_\mathrm{r}$. Ruess and Summers proved the following. Suppose f is asymptotically almost periodic with

$$F(t) := \int_0^t f(s)\mathrm{d}s \quad \text{bounded,}$$

and the improper Riemann integral of f_s exists in E. If E does not contain c_0 then F is asymptotically almost periodic. For details and discussion we refer to [8, Sec. 2.2]. As we see, a Jacobs–de Leeuw–Glicksberg type decomposition plays an essential role here.

Our main result, Corollary 12, provides the solution to the very first problem concerning power-bounded operators in this spirit. It contains the mentioned special case of Bolis' result when T is doubly power-bounded. For stating the result we first need some preparations, explaining an analogue of the decomposition above used by Ruess and Summers. Let E be a Banach space and let $T \in \mathscr{L}(E)$, which is from now on always assumed to be power-bounded. A vector $x \in E$ is called *asymptotically almost periodic* (a.a.p. for short) with respect to T if the (forward) orbit

$$\big\{T^n x : n \in \mathbb{N}\big\} \subseteq E$$

is relatively compact. Denote by E_{aap} the collection of a.a.p. vectors, which is a closed T-invariant subspace of E. We shall need the following form of the Jacobs–de Leeuw–Glicksberg decomposition for operators with relatively compact (forward) orbits; see [5, Chapter 16], or [6, Thm V.2.14] where the proof is explained for continuously parametrized semigroups instead of semigroups of the form $\{T^n : n \in \mathbb{N}\}$ (the proof is nevertheless the same).

Theorem 2 (Jacobs–de Leeuw–Glicksberg)**.** *Let E be a Banach space and let $T \in \mathscr{L}(E)$ have relatively compact orbits (T is hence power-bounded). Then there is a projection $P \in \mathscr{L}(E)$ commuting with T such that*

$$E_{\mathrm{r}} := \operatorname{rg} P = \big\{x \in E : Tx = \lambda x \textit{ for some } \lambda \in \mathbb{C},\ |\lambda| = 1\big\},$$
$$E_{\mathrm{s}} := \operatorname{rg}(I - P) = \ker P = \big\{x \in E : T^n x \to 0 \textit{ for } n \to \infty\big\}.$$

The restriction of T to E_{r} is a doubly power-bounded operator.

Note that the occurring subspaces and the projection depend on the linear operator T and—for the sake of better legibility, we chose not to reflect this dependence in notation. Now, we can apply this decomposition to a given power-bounded $T \in \mathscr{L}(E)$, or more precisely to the restriction of T to E_{aap}. We therefore obtain a decomposition

$$E_{\mathrm{aap}} = \operatorname{rg} P \oplus \ker P = E_{\mathrm{r}} \oplus E_{\mathrm{s}},$$

note that $E_{\mathrm{r}}, E_{\mathrm{s}} \subseteq E_{\mathrm{aap}}$; E_{s} is called the stable while E_{r} the reversible subspace. On $\operatorname{rg} P$ the restriction of T is a doubly power-bounded, and

$$\big\{T^n|_{E_{\mathrm{r}}} : n \in \mathbb{Z}\big\}$$

is a strongly compact group of operators.

Remark 3. Now suppose that $T \in \mathscr{L}(E)$ is even doubly power-bounded Then in the above decomposition $E_{\mathrm{s}} = \{0\}$ must hold. So if $x \in E_{\mathrm{aap}}$, then even the backward orbit is relatively compact, i.e.

$$\big\{T^n x : n \in \mathbb{Z}\big\} \quad \text{is relatively compact.}$$

A vector $x \in E_{\mathrm{r}}$ is also called *almost periodic.*

Lemma 1 tells that for $n, m \in \mathbb{N}$ with $m \geq n$ we have $(T^m - T^n)x \in E_{\mathrm{aap}}$ whenever $(T - I)x \in E_{\mathrm{aap}}$.

We are interested in whether $x \in E_{\mathrm{aap}}$ if $(T - I)x \in E_{\mathrm{aap}}$. The answer would be trivially "yes", if we knew that $T^n x - x$ actually converges as $n \to \infty$. Here is a slightly more complicated view on trivial fact:

Remark 4. a) Suppose we know $x \in E_{\mathrm{aap}}$. Then we can apply $I-P$ to x and obtain

$$(I-P)(T^n - I)x = (I-P)(T^n x - x) = T^n(I-P)x - (I-P)x \to (P-I)x$$

for $n \to \infty$. Hence if $x \in E_{\mathrm{aap}}$, then $(I-P)(T^n - I)x$ must be convergent.

b) Suppose that $(I-P)(T^n-I)x$ converges for $n \to \infty$ (note again that $T^n x - x \in E_{\mathrm{aap}}$, so we can apply the projection $I-P$ to it). If $(T-I)x \in E_{\mathrm{aap}}$ belongs even to the stable part, then $(T^n - I)x \in E_{\mathrm{s}}$, so $(I-P)(T^n-I)x = (T^n - I)x$. Hence $T^n x - x$ converges as $n \to \infty$ by assumption, implying $x \in E_{\mathrm{aap}}$.

It remains to study the case when $(T-I)x \in E_{\mathrm{r}} = \operatorname{rg} P$. The next is a preparatory lemma.

Lemma 5. *Suppose x is not an a.a.p. vector but $(T-I)x \in E_{\mathrm{aap}}$. Furthermore, suppose that $(I-P)(T^n x - x)$ converges. Then there is a $\delta > 0$ and a subsequence (n_k) of $\mathbb{N}$ such that*

$$\|P(T^{n_k}x - T^{n_\ell}x)\| \geq \delta \quad \textit{for all } k, \ell \in \mathbb{N},\ k \neq \ell.$$

Proof. By the non-a.a.p. assumption there is a subsequence (n_k) of $\mathbb{N}$ and a $\delta > 0$ such that $\|T^{n_k}x - T^{n_\ell}x\| > 2\delta$ for $\ell \neq k$. By the other assumption, however, $(I-P)(T^{n_k}x - x)$ is a Cauchy-sequence so when leaving out finitely many members, we can pass to a subsequence with $\|(I-P)(T^{n_k}x - T^{n_\ell}x)\| < \delta$ for all $k, \ell \in \mathbb{N}$, $\ell, k \geq k_0$. The assertion follows from this. $\square$

Note that in the situation of this lemma we necessarily have $\|P\| > 0$.

Lemma 6. *Let E be a Banach space, let $T \in \mathscr{L}(E)$ be power-bounded, and let $x_1, \dots x_m \in E$ be a.a.p. vectors. For every sequence $(n_k) \subseteq \mathbb{N}$ there is a subsequence (n'_k) with $n'_k - n'_{k-1} \to \infty$ and*

$$\|T^{n'_k}x_i - T^{n'_{k-1}}x_i\| \to 0 \quad \textit{for all } i = 1, \dots, m \textit{ as } k \to \infty.$$

Proof. Consider the Banach space $X = E^m$ and the diagonal operator $S \in \mathscr{L}(X)$ defined by $S(y_i) = (Ty_i)$. This is trivially power-bounded and $(x_i) \in X$ is an a.a.p. vector. The assertion follows from this, since $(S^{n_k}(x_i))$ has a Cauchy subsequence $(S^{n'_k}(x_i))$ with $n'_k - n'_{k-1} \to \infty$ as $k \to \infty$. $\square$

We now come to the answer of the initial question.

Theorem 7. *Let E be Banach space which does not contain an isomorphic copy of c_0, and let $T \in \mathscr{L}(E)$ be a power-bounded operator. If $x \in E$ and $(T-I)x$ is an a.a.p. vector with $(I-P)(T^n - I)x$ convergent for $n \to \infty$, then x itself is a.a.p. vector.*

The proof is by contradiction, i.e., we suppose that there is some $x \in E$ satisfying the assumptions of the theorem but being not asymptotically almost periodic. The contradiction arises then by finding a copy of c_0 in E, for which we shall use the classical result of Bessaga and Pełczyński [3] in the following form, see also [4, Thms. 6 and 8].

Theorem 8 (Bessaga–Pełczyński). *Let E be a Banach space and let $x_n \in E$ be vectors such that the partial sums are unconditionally bounded (i.e., $\sum_{j=1}^N x_{n_j}$ are uniformly bounded for all subsequences (n_j) of $\mathbb{N}$) and such that the series $\sum x_i$ is nonconvergent. Then E contains a copy of c_0.*

The idea of the proof is based on Basit's paper, but it is not a direct modification, since we do not know whether we can apply the projections P to x or to Tx.

Proof of Theorem 7. We argue indirectly. Assume that $x \notin E_{\mathrm{aap}}$, so by Lemma 5 we can take a subsequence (n_k) and a $\delta > 0$ such that $\|P(T^{n_k}x - T^{n_\ell}x)\| > \delta$ for all $k, \ell \in \mathbb{N}$ with $k \neq \ell$. Next we construct a sequence that fulfills the conditions of the Bessaga–Pelczynski Theorem 8, hence exhibiting a copy of c_0 in E. First of all let $M := \max(\sup\{\|T^n|_{E_{\mathrm{r}}}\| : n \in \mathbb{Z}\}, \sup\{\|T^n\| : n \in \mathbb{N}\}, \|P\|)$. Take $k_1 \in \mathbb{N}$ such that $\|P(T^{k_1}x - x)\| > \delta/M$ (use Lemma 5), and suppose that the strictly increasing finite sequence k_i, $i = 1, \dots, m$ is already chosen. For a subset $F \subset \{1, 2, \dots, m\}$ denote by ΣF the sum $\sum_{i \in F} k_i$ (if $F = \emptyset$, then $\Sigma F = 0$). Each of the finitely many vectors $T^{\Sigma F}x - x$ belongs to E_{aap} by Lemma 1. By using Lemma 6 we find $k, \ell \in \mathbb{N}$ with $k - \ell > k_m$ such that

$$\big\|T^{n_k}(T^{\Sigma F}x - x) - T^{n_\ell}(T^{\Sigma F}x - x)\big\| \leq \frac{1}{M2^m} \quad \text{for all } F \subseteq \big\{1, \dots, m\big\}.$$

By setting $k_{m+1} := n_k - n_\ell$ we obtain

$$\big\|T^{k_{m+1}}P(T^{\Sigma F}x - x) - P(T^{\Sigma F}x - x)\big\| \leq \frac{1}{2^m} \tag{1}$$

for all $F \subseteq \{1, \dots, m\}$. We also have

$$M\big\|P(T^{k_{m+1}}x - x)\big\| \geq \big\|T^{n_\ell}P(T^{k_{m+1}}x - x)\big\| = \big\|P(T^{n_k}x - T^{n_\ell}x)\big\| \geq \delta,$$

and hence we obtain

$$\big\|P(T^{k_{m+1}}x - x)\big\| \geq \frac{\delta}{M}.$$

Let $x_i := P(T^{k_i}x - x)$. We claim that the sequence (x_i) fulfills the conditions of Theorem 8. Indeed, we have $\|x_i\| \geq \delta/M$ by construction so the series $\sum x_i$ cannot be convergent. For $m \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_m$ we have

$$\begin{aligned}
-\sum_{j=1}^{m} x_{i_j} = \;& T^{k_{i_m}}P\Big(T^{k_{i_1}+\cdots+k_{i_{m-1}}}x - x\Big) - P\Big(T^{k_{i_1}+\cdots+k_{i_{m-1}}}x - x\Big) \\
& + T^{k_{i_{m-1}}}P\Big(T^{k_{i_1}+\cdots+k_{i_{m-2}}}x - x\Big) - P\Big(T^{k_{i_1}+\cdots+k_{i_{m-2}}}x - x\Big) \\
& \vdots \\
& + T^{k_{i_2}}P\Big(T^{k_{i_1}}x - x\Big) - P\Big(T^{k_{i_1}} - x\Big) \\
& + P(x - T^{k_{i_1}+\cdots+k_{i_m}}x).
\end{aligned}$$

By (1) we obtain

$$\Big\|\sum_{j=1}^{m} x_{i_j}\Big\| \leq \sum_{j=2}^{m} \frac{1}{2^{i_j - 1}} + M\|x\| + M^2\|x\| \leq M' < +\infty.$$

It follows that E contains a copy of c_0, a contradiction. □

If T is doubly power-bounded, then by Remark 3 we have $E_{\mathrm{s}} = \{0\}$, and hence $(I - P) = 0$. So we obtain the following special case of Basit's more general result:

Corollary 9 (Basit)**.** *Let E be Banach space E which does not contain a copy of* c_0*, and let $T \in \mathscr{L}(E)$ be a doubly power-bounded operator. If $x \in E$ and $(T-I)x$ is an a.a.p. vector, then so is x itself.*

The above results are certainly not valid for arbitrary Banach spaces. A counterexample is actually provided by the very same one showing that the analogue of the Bohl–Bohr theorem fails for arbitrary Banach-valued functions, see [7] or [9, Sec. 2.1].

Example 10. Consider $E = \mathrm{BUC}(\mathbb{R};\mathrm{c}_0)$, and T the shift by $a > 0$, and $x(t) := (\sin \frac{t}{2^n})_{n\in\mathbb{N}}$. Then $T \in \mathscr{L}(E)$ is doubly power-bounded, and we have

$$\left|\sin \tfrac{t+h}{2^n} - \sin \tfrac{t}{2^n}\right| = \left|\sin \tfrac{h}{2^{n+1}} \cos \tfrac{2t+h}{2^{n+1}}\right| \leq \varepsilon$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ if $|h|$ is sufficiently small, therefore $x \in \mathrm{BUC}(\mathbb{R};\mathrm{c}_0)$. On the other hand x is not an a.a.p. vector since the set

$$\left\{(\sin \tfrac{t+ma}{2^n})_{n\in\mathbb{N}} : m \in \mathbb{N}, t \in \mathbb{R}\right\} \subseteq \mathrm{c}_0$$

is not relatively compact. On the other hand,

$$y(t) := [(T-I)x](t) = (\sin \tfrac{t+a}{2^n} - \sin \tfrac{t}{2^n})_{n\in\mathbb{N}} = (\sin \tfrac{a}{2^{n+1}} \cos \tfrac{2t+a}{2^{n+1}})_{n\in\mathbb{N}}$$

is almost periodic, because $y(t)_n \to 0$ uniformly in $t \in \mathbb{R}$ as $n \to \infty$.

Next we show that on c_0 itself there is bounded linear operator satisfying the assumptions of Theorem 7 but for which the conclusion of that theorem fails to hold.

Example 11. It suffices to exhibit an example on $E := \mathrm{c}$. Let $(a_n) \in \mathrm{c}$ be a sequence with $|a_n| = 1$ for all $n \in \mathbb{N}$ and $a_n \neq b := \lim_{n\to\infty} a_n$ for all $n \in \mathbb{N}$. Define

$$T(x_n) := (a_n x_n).$$

Then $T \in \mathscr{L}(E)$ with $\|T\| = 1$. Moreover, T is invertible and doubly power-bounded. Since the standard basis vectors of c_0 are eigenvectors of T corresponding to unimodular eigenvalues, they all belong to E_{r}, and hence $\mathrm{c}_0 \subseteq E_{\mathrm{r}} \subseteq E_{\mathrm{aap}}$. Moreover, since T is doubly power-bounded we have $E_{\mathrm{s}} = \{0\}$, $I - P = 0$, and the condition "$(I-P)(T^n x - x)$ converges" is trivially satisfied for every $x \in E$. Not all vectors are a.a.p. with respect to T. It suffices to prove this for the case when $b = \lim_{n\to\infty} a_n = 1$, otherwise we can pass to the operator $b^{-1}T$, which has precisely the same a.a.p. vectors as T. Now suppose by contradiction that $E = E_{\mathrm{aap}}$ holds. Then T is mean ergodic on E, which is equivalent to the fact that $\ker(T-I)$ separates $\ker(T'-I)$, see, e.g., [5, Ch. 8]. But this is false, as $\dim\ker(T-I) = 0$ and $\dim\ker(T'-I) \geq 1$. Hence $E \neq E_{\mathrm{aap}}$ and it also follows that $E_{\mathrm{r}} = E_{\mathrm{aap}} = \mathrm{c}_0$.

Finally, we indeed suppose $b = 1$. Then, since $\mathrm{ran}(T-I) \subseteq \mathrm{c}_0 = E_{\mathrm{aap}}$, we obtain that $(T-I)x$ is a.a.p., for every $x \in E$, but not all $x \in E$ belongs to E_{aap}.

By [1, Sec. 2.5] if c_0 is a closed subspace in a separable Banach space, then it is complemented in there. Thus Example 11 in combination with Theorem 7 yields the following:

Corollary 12. *For a separable Banach space E the following assertions are equivalent:*

(i) The Banach space E does no contain a copy of c_0*.*

(ii) For every power-bounded linear operator $T \in \mathscr{L}(E)$ and $x \in E$ the orbit

$$\{T^n x : n \in \mathbb{N}\} \subseteq E$$

is relatively compact if and only if

$$\{T^{n+1}x - T^n x : n \in \mathbb{N}\} \subseteq E$$

is relatively compact and $(I-P)(T^n x - x)$ is convergent for $n \to \infty$.

We close this paper by two consequences of the previous results, interesting in their own right:

Corollary 13. *Let E be Banach space not containing c_0. Then for every $x \in E$, $T \in \mathscr{L}(E)$ doubly power-bounded operator and $m \in \mathbb{N}$, $m \geq 1$, the relative compactness of the two sets*

$$D_1 := \left\{T^{n+1}x - T^n x : n \in \mathbb{N}\right\} \subseteq E$$

and

$$D_m := \left\{T^{n+m}x - T^n x : n \in \mathbb{N}\right\} \subseteq E$$

are equivalent.

Proof. If D_m is relatively compact, then so is $\{T^{nm+m}x - T^{nm}x : n \in \mathbb{N}\}$ and by Corollary 9 even $\{T^{nm}x : n \in \mathbb{N}\}$. By the continuity of T the set $B_k := \{T^{nm+k}x : n \in \mathbb{N}\}$ is relatively compact for all $k = 0, \ldots, m-1$. Since

$$\left\{T^n x : n \in \mathbb{N}\right\} = B_1 \cup B_2 \cup \cdots \cup B_{m-1},$$

the relative compactness of the (forward) orbit of x follows. But this implies the relative compactness of D_1. That the relative compactness of D_1 implies that of D_m, is true without any assumption on the Banach space E, see Lemma 1. □

Example 14. Let $E := \mathrm{c}$ and for $m \in \mathbb{N}$, $m \geq 2$ fixed let T be as in Example 10 with $\lim_{n\to\infty} a_n = b \in \mathbb{C}$ an m^{th} root of unity. Then we have $E_{\mathrm{aap}} = E_{\mathrm{r}} = \mathrm{c}_0$. Since $\mathrm{rg}\,(T^m - I) \subseteq \mathrm{c}_0$ and $\mathrm{rg}\,(T-I) \subseteq \mathrm{c}_0 + (b-1)\mathbf{1}$ ($\mathbf{1}$ is the constant 1 sequence), we obtain that for this doubly power-bounded operator T and for every $x \in E$ the set D_m as in Corollary 13 is compact, while D_1 is not.

Similarly as for Corollary 12, we obtain from Corollary 13 and Example 14 the next characterization.

Corollary 15. *A separable Banach space E does not contain a copy of c_0 if and only if for every $x \in E$, $T \in \mathscr{L}(E)$ doubly power-bounded operator and $m \in \mathbb{N}$ the compactness of the two sets*

$$\left\{T^{n+1}x - T^n x : n \in \mathbb{N}\right\}$$

and

$$\left\{T^{n+m}x - T^n x : n \in \mathbb{N}\right\}$$

are equivalent.

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